M. Math IInd year Back paper examination .
Advanced Functional analysis

Answer all the 10 questions. Each question is worth 10 points.

If you are using any result proved in the class, you need to state it correctly.

1. Let $X$ be a locally convex completely metrizable real topological vector space and let $A \subset X$ be a compact set. Show that the closed convex hull of $A$ is a compact set.

2. Let $X$ be a locally convex space and $Y \subset X$ be a closed proper subspace. Show that there is a non-zero $\Lambda' \in X^*$ such that $\Lambda'(Y) = 0$.

3. Let $(X, d)$ be a metrizable real topological vector space. Let $T : X \to R$ be a linear map such that for any $x_n \to 0$ in $X$, $\{T(x_n)\}$ is a bounded sequence of real numbers. Show that $ker(T)$ is a closed set.

4. State and prove the Banach-Alaoglu theorem in locally convex topological vector spaces.

5. Let $K$ be a compact convex subset of a locally convex topological vector space. Let $\partial_e K$ denote the set of all extreme points of $K$. Let $f : K \to K$ be a continuous affine and onto map. For $k_0 \in \partial_e K$ show that $f^{-1}(k_0)$ is an extreme closed convex set.

6. Let $X, Y$ be metrizable topological vector spaces and $X$ is a complete metric space, $Y$ is of second category. Let $\Lambda : X \to Y$ be a continuous linear onto map. Show that $Y$ is also a complete metric space.

7. Let $\Gamma$ denote the unit circle. Let $f : [0, 1] \to \Gamma$ be a continuous map. Show that $f$ is an extreme point of the closed unit ball of $C([0, 1])$.

8. State and prove the Pettis' measurability theorem.

9. Let $X$ be a Banach space and let $f : [0, 1] \to X$ be a strongly measurable function. Suppose $x^* \circ f = 0$ a.e for all $x^* \in X^*$. Is it true that $f = 0$ a.e? Justify your answer.

10. Let $K$ be a compact convex set in a locally convex topological vector space. Let $\mu$ be a regular Borel probability measure on $K$. Show that there is a $x_0 \in K$ such that $x^*(x_0) = \int_K x^* d\mu$ for all $x^* \in X^*$.